Three Extensions of the Basic Unit-Level Model

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Outline

- Introduction
    - Linear, unit-level model
- Extension 1: Lognormal models
    - Empirical best prediction concepts
- Extension 2: Zero-inflated lognormal models
    - Population-level covariates
- Extension 3: Informative sampling
    - Nonlinear parameters, exponential dispersion families

What is small area estimation?

- Large-scale surveys play an important role in the federal statistical system
  - National Crime Victimization Survey – criminal victimization rates for individuals ages 12 and older in the US
  - National Resources Inventory – characteristics related to natural resources and agriculture
  - Canadian labor force survey – parameters related to employment

What is small area estimation?

- Complex surveys are often designed to produce estimates for *large* estimation domains. Data users often request estimates for estimation domains with *small* sample sizes
  - National Crime Victimization Survey publishes national level estimates
    - State-level estimates are of interest
  - National Resources Inventory publishes state-level estimates
    - County-level estimates are of interest
  - Canadian labor force survey produces estimates for broad employment categories at the provincial level
    - Detailed employment categories are of interest

What is Small Area Estimation?

- The challenge
  - As a result of small sample sizes, direct estimators are unreliable
- The solution: small area estimation (Rao & Molina 2015)
  - Use models to obtain more efficient estimates

Basic Unit-Level Model: Set-up

- $i = 1, \ldots, D$ index the areas
- $j = 1, \ldots, N_i$ index the elements in the entire population for area i
- $j = 1, \ldots, n_i < N_i$ index the elements in the sample for area i
- y_{ij} is the variable of interest for unit j in area i
- The parameter of interest is the area mean defined by

$$\theta_i = \frac{1}{N_i} \sum_{j=1}^{N_i} y_{ij}$$

- Data available

$$\{y_{ij} : j = 1, \ldots, n_i\} \cup \{\boldsymbol{x}_{ij} : j = 1, \ldots, n_i\} \cup \{\bar{\boldsymbol{x}}_{N,i} : i = 1, \ldots, D\}$$

$$\bar{\boldsymbol{x}}_{N,i} = \frac{1}{N_i} \sum_{j=1}^{N_i} \boldsymbol{x}_{ij}$$

Basic Unit-Level Model (Battese et al. 1988)

- Model Assumption

$$
\begin{aligned}
y_{ij} &= \beta_0 + \boldsymbol{x}_{ij}'\boldsymbol{\beta}_1 + u_i + e_{ij}, \\
u_i &\overset{iid}{\sim} N(0, \sigma_u^2) \\
e_{ij} &\overset{iid}{\sim} N(0, \sigma_e^2)
\end{aligned}
$$

- Use REML (R function lmer in lme4 package, or SAE R package) to obtain estimates: $\hat{\beta}_0$, $\hat{\boldsymbol{\beta}}_1$, $\hat{\sigma}_u^2$, $\hat{\sigma}_e^2$

Basic Unit-Level Model

Challenges

- Skewed response variables
- Zero-inflated data
- Informative sampling

Part 1: Unit-Level Lognormal Model (joint work with Hukum Chandra)

- Small area model for skewed data
- Illustrate basic concepts of small area prediction under nonlinear models

Unit-Level Lognormal Model: Motivation

- Response variable (Y) has a non-normal distribution
  - Skewed
  - Positive support
  - Variance increases with the mean
  - Nonlinear associations to covariates
- Linear predictors inefficient

Unit-Level Lognormal Model: Framework

- Areas: $i = 1, \ldots, D$; units: $j = 1, \ldots, N_i$

$$\log(y_{ij}) = \boldsymbol{z}_{ij}'\boldsymbol{\beta} + u_i + e_{ij}$$
$$(u_i, e_{ij}) \overset{iid}{\sim} N(\mathbf{0}, \mathrm{diag}(\sigma_u^2, \sigma_e^2))$$
$$\boldsymbol{\theta} = (\boldsymbol{\beta}', \sigma_u^2, \sigma_e^2)', \quad \hat{\boldsymbol{\theta}} = (\hat{\boldsymbol{\beta}}', \hat{\sigma}_u^2, \hat{\sigma}_e^2)'$$

- Data available

$$\{y_{ij} : j = 1, \ldots, n_i\} \cup \{\boldsymbol{z}_{ij} : j = 1, \ldots, N_i\}, i = 1, \ldots, D$$
$$\boldsymbol{y}_s = \{y_{ij} : j = 1, \ldots, n_i, i = 1, \ldots, D\}$$

  - n_i = sample size, N_i = population size

- Quantity to predict: small area mean

$$\bar{y}_{N_i} = \frac{1}{N_i}\left[\sum_{j=1}^{n_i} y_{ij} + \sum_{j=n_i+1}^{N_i} y_{ij}\right]$$

Unit-Level Lognormal Model: Predictor

- Best (Bayes) predictor (minimum MSE) – general

$$\bar{y}^{B}_{N_i}(\boldsymbol{\theta}) = \frac{1}{N_i}\left\{\sum_{j=1}^{n_i} y_{ij} + \sum_{j=n_i+1}^{N_i} E[y_{ij} \mid \boldsymbol{\theta}, \boldsymbol{y}_s]\right\}$$

- Conditional expectation for the lognormal

$$E[y_{ij} \mid \boldsymbol{\theta}, \boldsymbol{y}_s] = \exp\left[\boldsymbol{z}'_{ij}\boldsymbol{\beta} + \gamma_i(\bar{\ell}_{si} - \bar{\boldsymbol{z}}'_{si}\boldsymbol{\beta}) + \frac{\sigma_e^2}{2}\left(\frac{\gamma_i}{n_i} + 1\right)\right]$$

$$\bar{\ell}_{si} = n_i^{-1}\sum_{j=1}^{n_i}\log(y_{ij}), \quad \gamma_i = \sigma_u^2(\sigma_u^2 + n_i^{-1}\sigma_e^2)^{-1}$$

- Justification for $E[y_{ij} \mid \boldsymbol{\theta}, \boldsymbol{y}_s]$

$$\log(y_{ij}) \mid \boldsymbol{\theta}, \boldsymbol{y}_s \sim N(\boldsymbol{z}'_{ij}\boldsymbol{\beta} + \gamma_i(\bar{\ell}_{si} - \bar{\boldsymbol{z}}'_{si}\boldsymbol{\beta}), \gamma_i\sigma_e^2 n_i^{-1} + \sigma_e^2)$$

Unit-Level Lognormal Model: Predictor

- Empirical best predictor (general)

$$\hat{y}_{N_i}^{EB} = \bar{y}_{N_i}^{MMSE}(\hat{\boldsymbol{\theta}}) = \frac{1}{N_i}\left\{\sum_{j=1}^{n_i} y_{ij} + \sum_{j=n_i+1}^{N_i} E[y_{ij} \mid \hat{\boldsymbol{\theta}}, \boldsymbol{y}_s]\right\}$$

- $\hat{y}_{N_i}^{EB}$ for the lognormal biased due to nonlinear transformation of $\hat{\boldsymbol{\theta}}$

$$E\left\{E[y_{ij} \mid \hat{\boldsymbol{\theta}}, \boldsymbol{y}_s] - E[y_{ij} \mid \boldsymbol{\theta}, \boldsymbol{y}_s]\right\} \neq 0$$

- Multiplicative bias correction
  - Bias-corrected estimator - $\hat{y}_{N_i}^{EB.BC}$

Unit-Level Lognormal Model: MSE Estimation

- General MSE of an EB predictor

$$
\begin{aligned}
MSE(\hat{y}_{N_i}^{EB}) &= E[(\hat{y}_{N_i}^{EB} - \bar{y}_{N_i})^2] \\
&= \underbrace{E[(\bar{y}_{N_i}^{B}(\boldsymbol{\theta}) - \bar{y}_{N_i})^2]}_{M_{1i}(\boldsymbol{\theta})} + \underbrace{E[(\hat{y}_{N_i}^{EB} - \bar{y}_{N_i}^{B}(\boldsymbol{\theta}))^2]}_{M_{2i}(\boldsymbol{\theta})}
\end{aligned}
$$

$$
\begin{aligned}
M_{1i}(\boldsymbol{\theta}) &= E[V(\bar{y}_{N_i} \mid \boldsymbol{y}_s)] \\
&= \text{ MSE of best predictor constructed with (unknown)} \\
&\quad \text{true parameters} \\
M_{2i}(\boldsymbol{\theta}) &= \text{variance due to estimation of } \boldsymbol{\theta}
\end{aligned}
$$

Unit-Level Lognormal Model: MSE Estimation

- Closed-form expression for $M_{1i}(\boldsymbol{\theta})$

$$
\begin{aligned}
M_{1i}(\boldsymbol{\theta}) &= MSE\{\bar{y}^B_{N_i}(\boldsymbol{\theta})\} \\
&= \frac{\kappa_i}{N_i^2}\left[(\sum_{j\in\bar{s}_i}\exp(\boldsymbol{z}'_{ij}\boldsymbol{\beta}))^2\xi_i + (\sum_{j\in s_i}\exp(2\boldsymbol{z}'_{ij}\boldsymbol{\beta}))\psi_i\right]
\end{aligned}
$$

$(\xi_i, \psi_i, \kappa_i)$ are known functions of $\sigma_u^2, \sigma_e^2, n_i$

- Taylor series approximation for $M_{2i}(\boldsymbol{\theta})$

Unit-Level Lognormal Model: MSE Estimation

- Plug-in estimator

$$\hat{MSE}_{1i} = M_{1i}(\hat{\boldsymbol{\theta}}) + \hat{M}_{2i}$$

  - Biased because $E[M_{1i}(\hat{\boldsymbol{\theta}}) - M_{1i}(\boldsymbol{\theta})] \neq 0$

- Bias-reduced estimator:

$$\hat{MSE}_{2i} = M_{1i}(\tilde{\boldsymbol{\theta}}_i) + \hat{M}_{2i}$$

  - $\tilde{\boldsymbol{\theta}}_i$ depends on Taylor expansion of $M_{1i}(\boldsymbol{\theta})$
  - $E[M_{1i}(\tilde{\boldsymbol{\theta}}_i) - M_{1i}(\boldsymbol{\theta})] \approx 0$
  - The bias-adjusted MSE estimator is non-negative

Unit-Level Lognormal: Simulation Models

- $N = 9990, D = 30$

$$\log(y_{ij}) = \beta_0 + \beta_1 z_{ij} + u_i + e_{ij}$$
$$(z_{ij}, u_i, e_{ij}) \sim N\{(\mu_z, 0, 0), \text{diag}(\sigma_z^2, \sigma_u^2, \sigma_e^2)\}$$

Four Parameter Sets

σ_z^2	1.6	1.6	1.2	1.2
$\sigma_u^2\sigma_e^{-2}$	0.5	0.2	0.5	0.2
σ_u	0.6	0.4	0.7	0.5

$(\mu_z, \beta_0, \beta_1) = (3.253, -1.62, 0.9)$

- Mean and variance of y_{ij} approximately equal to mean and variance of the number of chickens per segment in a 1960 USDA area survey (Fuller, 1991)

Simulations: Designs and Estimators

- MC sample size of 2000
- For each MC sample,
  1. Generate a new set of z_{ij}
  2. Select a stratified SRS with areas as strata
     - $n_i N_i^{-1} \approx 0.0375$
     - $n_i = 5,\ i = 1, \ldots, 15;\ n_i = 20,\ i = 16, \ldots, 30$
  - Estimators
     - TrIP - Indirect predictor based on Karlberg (2000)
     - TrMBD - Model-based direct estimator of Chandra and Chambers (2011)
     - EB - empirical best predictor
     - EB.BC - EB predictor with multiplicative bias correction

Simulations: Results

- Relative bias of predictor, $\hat{y}_{N_i}$

$$RB_i = \frac{E_{MC}[\hat{y}_{N_i} - \bar{y}_{N_i}]}{E[\bar{y}_{N_i}]}$$

- TrIP and TrMBD unibased
- RB_i of EB larger for $\sigma_z = 2$ than $\sigma_z = 1.6$
  - For $\sigma_z = 1.2$ and $n_i = 5$, the average RB_i is 1.3 for $\sigma_u = 0.5$ and 1.4 for $\sigma_u = 0.2$
  - Average RB_i less than 3% of MC RMSE
  - EB.BC unbiased
  - RB_i smaller for $n_i = 20$ than $n_i = 5$

Simulations: Results

- MSE of predictor, $\hat{y}_{N_i}$, relative to MSE of EB.BC predictor

$$RelMSE_i = \frac{MSE_{MC}(\hat{y}_{N_i})}{MSE_{MC}(\hat{y}_{N_i}^{EB.BC})}$$

		Average $RelMSE_i$			
		$n_i = 5$		$n_i = 20$	
$\sigma_u^2 \sigma_e^{-2}$	σ_z	TrIP	trMBC	TrIP	TrMBD
0.5	1.6	2.2	1.8	6.1	1.5
0.2	1.6	1.3	2.8	2.5	1.8
0.5	1.2	2.2	2.8	5.9	2.2
0.2	1.2	1.3	4.5	2.5	2.6

Simulations: Results

- Properties of MSE estimators
- Relative bias

$$RB_i = \frac{E_{MC}[\hat{MSE}_{2i}] - MSE_{MC}(\hat{y}_{N_i}^{EB.BC})}{MSE_{MC}(\hat{y}_{N_i}^{EB.BC})}$$

  - Average RB_i between -2.2% and 10.7%
- Coverage of normal theory CI's with nominal coverage of 95%
  - Empirical coverage between 94.6% and 95.3%

Unit-Level Lognormal: Take-Home Messages

- Unit-Level lognormal model
  - Extends the linear, unit-level model to handle skewed, positive response variables
- Empirical Bayes predictor
  - Closed-form expression
  - More efficient than competitors in simulations
- MSE estimator
  - Closed-form expression
  - Relative biases less than 11%
  - Empirical coverages close to the nominal level

Unit-Level Lognormal: Take-Home Messages

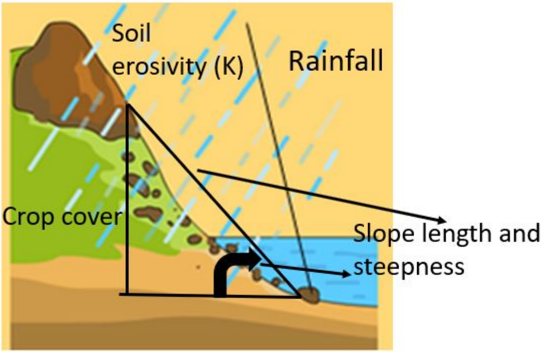
- Basic concepts of SAE for nonlinear models
  - EB predictor is estimate of conditional expectation of small area mean given data
  - MSE of EB predictor decomposes into a sum of two terms
    - Leading term = conditional variance of small area mean given data
    - Second term = variance due to estimation of fixed parameters

Part 2: Zero-Inflated Lognormal Model (joint work with Annie Lyu)

- Extends the lognormal to handle zero-inflated data
- Apply the method to data from an agricultural survey
- Discuss the challenges and importance of obtaining unit-level auxiliary information at the population level

Sheet and Rill Erosion

- Sheet and rill erosion (SRE) – transport of soil from thin surface layers (sheets) or small channels (rills) due to rainfall or shallow runoff



Factors Impacting Erosion

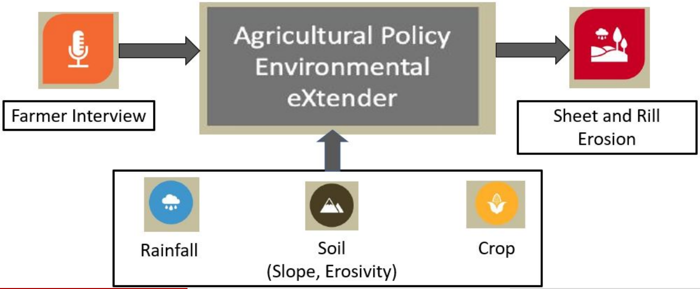
- Rainfall
- Soil properties
  - Slope length/steepness
  - Erosivity (ease of detachment)
- Crop managements
- Conservation practices

- SRE degrades agricultural land and pollutes water
- Conservation policies rely on estimates of SRE

Small Area Estimation for SRE

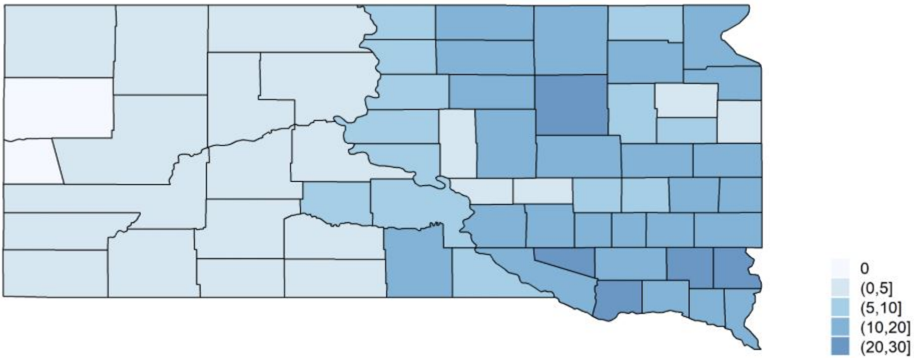
Conservation Effects Assessment Project (CEAP)

- Two-phase survey that quantifies water & wind erosion on cropland
  - CEAP conducts farmer interviews at a subset of locations classified as cropland in a larger survey called the National Resources Inventory (NRI)
- Survey data and auxiliary info. on soils and climate are processed through the APEX computer model
- An approximation for SRE is one APEX output



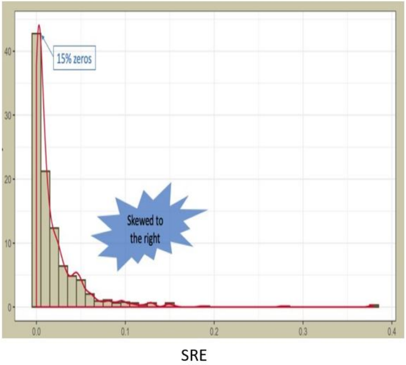
Small Area Estimation for SRE

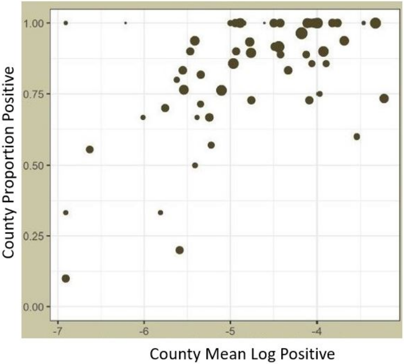
- Estimates of average SRE in South Dakota counties are of interest
  - CEAP county sample sizes are small $\longrightarrow$ small area estimation



Small Area Estimation for SRE

- Exploratory analysis of CEAP SRE data





Small Area Estimation for SRE

Zero-Inflated Lognormal Model

$$\text{SRE: } y^*_{ij} = y_{ij}\delta_{ij}$$
$$y_{ij} > 0; \quad \delta_{ij} \sim \text{Bernoulli}(p_{ij})$$
$$i = 1, \ldots, 64 \text{ (SD Counties)}; \quad j = 1, \ldots, N_i \text{ (crop field in pop.)}$$

Positive Part

$$\log(y_{ij}) = \beta_0 + \boldsymbol{z}'_{1,ij}\boldsymbol{\beta}_1 + u_i + e_{ij}$$
$$e_{ij} \overset{iid}{\sim} N(0, \sigma_e^2)$$

Binary Part

$$\text{logit}(p_{ij}) = \alpha_0 + \boldsymbol{z}'_{2,ij}\boldsymbol{\alpha}_1 + b_i$$
$$p_{ij} := p_{ij}(b_i)$$

Correlation b/ Positive and Binary Parts

$$\begin{pmatrix} u_i \\ b_i \end{pmatrix} \overset{iid}{\sim} \text{BVN}(\mathbf{0}, \boldsymbol{\Sigma}_{ub}), \quad \boldsymbol{\Sigma}_{ub} = \begin{pmatrix} \sigma_u^2 & \sigma_{ub} \\ \sigma_{ub} & \sigma_b^2 \end{pmatrix}$$
$$\sigma_{ub} = \rho\sigma_u\sigma_b$$

Small Area Estimation for SRE

- County mean of interest: $\bar{y}^*_{N_i} = N_i^{-1} \sum_{j=1}^{N_i} y^*_{ij}$
- Data for small area prediction:
  - $(\boldsymbol{y}^*, \boldsymbol{z}) = \{y^*_{ij} : j \in s_i\} \cup \{\boldsymbol{z}_{ij} = (\boldsymbol{z}'_{1,ij}, \boldsymbol{z}'_{2,ij})' : j \in s_i \cup \bar{s}_i\}$
  - s_i is sample for county i with sample size $n_i = |s_i|$
  - $\bar{s}_i$ is set of nonsampled elements in county i with $|\bar{s}_i| = N_i - n_i$
- Minimum MSE (Bayes) predictor:

$$\hat{\bar{y}}^{*\mathrm{MMSE}}_{N_i}(\boldsymbol{\theta}) = N_i^{-1}\left[\sum_{j\in s_i} y^*_{ij} + \sum_{j\in\bar{s}_i} \mathrm{E}\{y^*_{ij} \mid (\boldsymbol{y}^*, \boldsymbol{z}); \boldsymbol{\theta}\}\right]$$

$$\boldsymbol{\theta} = (\beta_0, \boldsymbol{\beta}'_1, \alpha_0, \boldsymbol{\alpha}'_1, \sigma^2_e, \sigma^2_u, \sigma^2_b, \rho)'$$

- Challenge
  - Correlation parameter ρ introduces a need for integration over the bivariate distribution of (u_i, b_i)
- Approach
  - Transform bivariate integrals to univariate integrals

Small Area Estimation for SRE: Empirical Bayes Predictor

- Gauss-Hermite approximation to univariate integral
- Empirical Bayes (EB) predictor:

$$\hat{\bar{y}}_{N_i}^{*\mathrm{MMSE}}(\hat{\boldsymbol{\theta}}) = \frac{1}{N_i}\Big[\sum_{j\in s_i} y_{ij}^* + \sum_{j\in \bar{s}_i} \mathrm{E}\{y_{ij}^* \mid (\boldsymbol{y}^*, \boldsymbol{z}); \hat{\boldsymbol{\theta}}\}\Big]$$

  - Maximum likelihood estimator $\hat{\boldsymbol{\theta}}$

Small Area Estimation for SRE: MSE Estimator

- "One-step" estimator of $M_{1i}(\boldsymbol{\theta})$
  - Replace $\boldsymbol{\theta}$ with MLE $\hat{\boldsymbol{\theta}}$
  - Use Gauss-Hermite to approximate univariate integral

- $M_{2i}^{boot}(\hat{\boldsymbol{\theta}})$ = parametric bootstrap estimator of $M_{2i}(\boldsymbol{\theta})$

"Semi-boot" MSE estimator

$$\hat{MSE}_i = M_{1i}(\hat{\boldsymbol{\theta}}) + M_{2i}^{boot}(\hat{\boldsymbol{\theta}})$$

Small Area Estimation for SRE: Covariates

* Covariates measure factors impacting erosion and are known for the full population of cropland in South Dakota
- Rainfall
  - *logR*: log R-factor, a measure of long-term, average rainfall in a county
- Soils
  - *logS*: log of slope steepness factor at a unit's location
  - *logK*: log of soil K-factor (erodibility index) at a unit's location
    - Higher K-factors indicate greater erosivity – potential for detachment
- Crop type
  - We use crop classifications from the Cropland Data Layer (CDL), a satellite-derived landcover map with 30meter^2 resolution
    - *is.soybean* $= 1$ if location classified as soybeans; 0 otherwise
    - *is.sprwht* $= 1$ if location classified as spring wheat; 0 otherwise

Small Area Estimation for SRE: Estimates and CI's

- Specific covariates selected with step-wise AIC

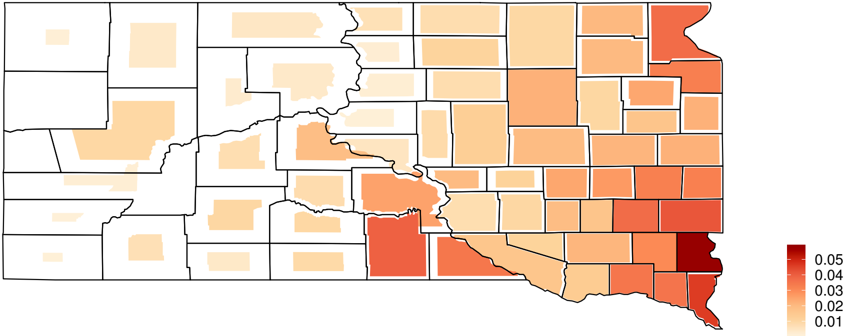
Maximum Likelihood Estimate & Bootstrap SE

	Positive Part		Binary Part	
	Estimate	(SE)	Estimate	(SE)
logR	2.19	(0.36)	4.94	(0.72)
logK	0.52	(0.23)		
logS	0.49	(0.08)	0.38	(0.21)
is.soybean			0.71	(0.33)
is.sprwht			0.98	(0.52)
Var: county	0.22		0.47	
Var: residual	1.23			

- Correlation $\hat{\rho} = 0.77$ with 95% bootstrap CI of $(0.21, 0.99)$

Small Area Estimation for SRE: Spatial Distribution of Predictions

- Cartogram of EB predictors; fraction of shaded area inversely proportional to CV



Discussion

- An understanding of sheet and rill erosion is important for conservation efforts
- Using CEAP data, we estimate mean SRE for South Dakota counties
  - Zero-inflated lognormal model
  - Non-trivial correlation betwen b_i and u_i
- A key challenge is deriving covariates that are available for the full population and relate to factors impacting erosion
  - We integrate NRI, CDL, and Soil Survey to obtain covariates for the population of interest

Part 3: Informative sampling (joint work with Abdulhakeem Eideh)

- Develop predictors for unit-level models under informative sampling
- Generalize procedures to the broad class of exponential dispersion families
- Prediction of nonlinear parameters
- Validate the methods through simulation and data analysis

Framework

- Areas: $i = 1, \ldots, D$ (all are sampled)
- Units: $j = 1, \ldots, N_i$
- Sample: $A_i \subset \{1, \ldots, N_i\}$
- Sample inclusion indicator: $I_{ij} = I[j \in A_i]$
- Probability that unit j in area i is selected: $\pi_{ij} = P_{ij}(I_{ij} = 1)$
- Weight: $w_{ij} = \pi_{ij}^{-1}$
- Covariate: $\boldsymbol{x}_{ij}$ observed for $j = 1, \ldots, N_i$
- Response: y_{ij} observed for $j \in A_i$

$$D_s = \{\boldsymbol{x}_{ij} : j \in U_i\} \cup \{y_{ij} : j \in A_i\} \cup \{w_{ij} : j \in A_i\}$$

Distributions (Pfeffermann & Sverchkov 2007)

Population Distributions

$$y_{ij} \overset{ind}{\sim} f_p(y_{ij} \mid u_i, \boldsymbol{x}_{ij}), j = 1, \ldots, N_i$$

$$u_i \overset{ind}{\sim} f_p(u_i \mid \theta_u, \phi_u), i = 1, \ldots, D$$

Sample Distributions

$$f_{si}(y_{ij} \mid \boldsymbol{x}_{ij}, u_i) = f_p(y_{ij} \mid u_i, \boldsymbol{x}_{ij}, I_{ij} = 1)$$

Complement Distribution

$$f_{ci}(y_{ij} \mid \boldsymbol{x}_{ij}, u_i) = f_p(y_{ij} \mid u_i, \boldsymbol{x}_{ij}, I_{ij} = 0)$$

Relationships

$$f_{ci}(y_{ij} \mid \boldsymbol{x}_{ij}, u_i) \propto E_s(w_{ij} - 1 \mid \boldsymbol{x}_{ij}, y_{ij}) f_{si}(y_{ij} \mid \boldsymbol{x}_{ij}, u_i)$$

$$f_p(y_{ij} \mid \boldsymbol{x}_{ij}, u_i) \propto E_s(w_{ij} \mid \boldsymbol{x}_{ij}, y_{ij}) f_{si}(y_{ij} \mid \boldsymbol{x}_{ij}, u_i)$$

Assumptions

Exponential Dispersion Family

$$
\begin{aligned}
f_{si}(y_{ij} \mid \theta_{ij}, \phi) &= \exp\left[\phi(y_{ij}\theta_{ij} - b(\theta_{ij})) + c(y_{ij}, \phi)\right] \\
\theta_{ij} &= g(\boldsymbol{x}_{ij}, u_i) \\
f(u_i \mid \theta_u, \phi_u) &= \exp\left[\phi_u(u_i\theta_u - b(\theta_u)) + c(u_i, \phi_u)\right]
\end{aligned}
$$

Models for the Weights (Pfeffermann & Sverchkov 2007, Kim & Wang 2023)

- Mean weight model

$$E_{si}(\pi_{ij}^{-1} \mid y_{ij}, \boldsymbol{x}_{ij}) = \exp(q_i + \gamma_1 y_{ij} + \boldsymbol{\gamma}_2' \boldsymbol{x}_{ij} y_{ij} + \boldsymbol{\gamma}_3' \boldsymbol{x}_{ij})$$

- Beta prime weight model: $\pi_{ij} \mid I_{ij} = 1 \overset{ind}{\sim} \text{Beta}(\mu_{ij}\phi_1 + 1, (1 - \mu_{ij})\phi_1)$

$$E_{si}(\pi_{ij}^{-1} - 1 \mid y_{ij}, \boldsymbol{x}_{ij}) = \alpha_{0,i}\exp(\alpha_1 y_{ij} + \boldsymbol{\alpha}_2' \boldsymbol{x}_{ij} y_{ij} + \boldsymbol{\alpha}_3' \boldsymbol{x}_{ij}) := \mu_{ij} - 1$$

Implications

Theorem 1

- Under the mean weight model,

$$f_p(y_{ij} \mid \boldsymbol{x}_{ij}, u_i) = \exp[\phi(y_{ij}\theta^*_{ij} - b(\theta^*_{ij})) + c(y_{ij}, \phi)]$$
$$\theta^*_{ij} = \theta_{ij} + \gamma_2/\phi + \boldsymbol{x}'_{1,ij}\boldsymbol{\gamma}_3/\phi$$

Theorem 2

- Under the beta-prime weight model,

$$f_{ci}(y_{ij} \mid \boldsymbol{x}_{ij}, u_i) = \exp[\phi(y_{ij}\tilde{\theta}_{ij} - b(\tilde{\theta}_{ij})) + c(y_{ij}, \phi)]$$
$$\tilde{\theta}_{ij} = \theta_{ij} + \alpha_2/\phi + \boldsymbol{x}'_{1,ij}\boldsymbol{\alpha}_3/\phi$$

Estimators

Max. Likelihood for $\boldsymbol{\psi}_1 = (\boldsymbol{\beta}', \phi_u, \theta_u)'$

$$\hat{\boldsymbol{\psi}}_1 = argmax_{\psi_1} \sum_{i=1}^{D} \log\left(\int_{-\infty}^{\infty} \left[\prod_{j=1}^{n_i} f_{si}(y_{ij} \mid \theta_{ij}(u), \phi)\right] f_s(u \mid \phi_u, \theta_u) du\right)$$

Least Squares for $\boldsymbol{\psi}_2 = (\gamma_2, \boldsymbol{\gamma}_3')'$

- Let $(\hat{q}_1, \ldots, \hat{q}_D, \hat{\gamma}_1', \hat{\gamma}_2, \hat{\gamma}_3')$ minimize $\sum_{i \in s} \sum_{j \in s_i} (\log(w_{ij}) - q_i - \boldsymbol{x}_{1,ij}' \gamma_1 - y_{ij}\gamma_2 - y_{ij}\boldsymbol{x}_{1,ij}'\gamma_3)^2$

Max. Likelihood for Beta-Prime Weight Model

$(\hat{\phi}_1, \hat{\boldsymbol{\alpha}}')' = argmax_{(\phi_1, \boldsymbol{\alpha}')'} L_3(\phi_1, \boldsymbol{\alpha})$, $\boldsymbol{\alpha} = (\alpha_{0,1}, \ldots, \alpha_{0,D}, \boldsymbol{\alpha}_1', \alpha_2, \boldsymbol{\alpha}_3')'$,

$$L_3(\phi_1, \boldsymbol{\alpha}) = \prod_{i,j} \frac{1}{\mathrm{B}(\mu_{ij}\phi_1 + 1, (1 - \mu_{ij})\phi_1)} \pi_{ij}^{\mu_{ij}\phi_1} (1 - \pi_{ij})^{(1-\mu_{ij})\phi_1 - 1}$$

Empirical Best Prediction Overview

- Generalizations of Molina & Rao (2010)
  - Simulate from complement distributions
- Algorithm 1: mean weight model
  - Simulate from population distribution
- Algorithm 2: beta-prime
  - Simulate from complement distribution

Empirical Best Prediction: Mean Weight Model

Algorithm 1: For $r = 1, \ldots, R$, repeat the following steps.

1. Generate $u_i^{(r)} \overset{iid}{\sim} f_i(u_i \mid D_s, \hat{\boldsymbol{\beta}}, \hat{\phi}, \hat{\phi}_u, \hat{\theta}_u)$ for $i = 1, \ldots, D$, where

$$f_i(u_i \mid D_s, \boldsymbol{\beta}, \phi, \phi_u, \theta_u) = \frac{[\prod_{j=1}^{n_i} f_{si}(y_{ij} \mid \theta_{ij}(u_i), \phi)] f_s(u_i \mid \theta_u, \phi_u)}{\int_{-\infty}^{\infty} [\prod_{j=1}^{n_i} f_{si}(y_{ij} \mid \theta_{ij}(u), \phi)] f_s(u \mid \theta_u, \phi_u) du}.$$

2. Generate $y_{ij}^{(r)} \overset{ind}{\sim} f_p(y_{ij} \mid \hat{\theta}_{ij}^{(r)}, \hat{\gamma}_2, \hat{\boldsymbol{\gamma}}_3)$, where $\hat{\theta}_{ij}^{(r)} = g(\boldsymbol{x}_{ij}' \hat{\boldsymbol{\beta}}, u_i^{(r)})$ for $j \in U_i$.
3. Define $\theta_i^{(r)}(\hat{\boldsymbol{\psi}}) = h(y_{i1}^{(r)}, \ldots, y_{iN_i}^{(r)})$.

Empirical best predictor: $\hat{\theta}_i = \tilde{\theta}_i(\hat{\boldsymbol{\psi}}) = R^{-1} \sum_{r=1}^{R} \theta_i^{(r)}(\hat{\boldsymbol{\psi}})$, $\hat{\boldsymbol{\psi}} = (\hat{\boldsymbol{\psi}}_1', \hat{\boldsymbol{\psi}}_2')'$

Empirical Best Prediction: Beta-Prime Weight Model

Algorithm 2: For $r = 1, \ldots, R$, repeat the following steps.

1. Generate $u_i^{(r)} \overset{iid}{\sim} f_i(u_i \mid D_s, \hat{\boldsymbol{\beta}}, \hat{\phi}, \hat{\phi}_u, \hat{\theta}_u)$ for $i = 1, \ldots, D$.
2. Generate $y_{ij}^{(r)} \overset{ind}{\sim} f_{ci}(y_{ij} \mid \hat{\theta}_{ij}^{(r)}, \hat{\alpha}_2, \hat{\boldsymbol{\alpha}}_3)$, where $\hat{\theta}_{ij}^{(r)} = g(\boldsymbol{x}_{ij}'\hat{\boldsymbol{\beta}}, u_i^{(r)})$ for $j \notin s_i$. For $j \in s_i$, set $y_{ij}^{(r)} = y_{ij}$.
3. Define $\theta_i^{(r)}(\hat{\boldsymbol{\psi}}) = h(y_{i1}^{(r)}, \ldots, y_{iN_i}^{(r)})$.

Empirical best predictor: $\hat{\theta}_i = \tilde{\theta}_i(\hat{\boldsymbol{\psi}}^{BP}; \boldsymbol{y}_{is}) = R^{-1} \sum_{r=1}^{R} \theta_i^{(r)}(\hat{\boldsymbol{\psi}}^{BP}; \boldsymbol{y}_{is})$,
$\hat{\boldsymbol{\psi}}^{BP} = (\hat{\boldsymbol{\psi}}_1', \hat{\alpha}_2, \hat{\boldsymbol{\alpha}}_3')'$

MSE Estimation

$$MSE(\hat{\theta}_i) = M_{1i} + M_{2i}$$
$$M_{1i} = E[V(\theta_i \mid D_s)]$$
$$M_{2i} \text{ reflects variation of parameter estimators}$$

Mean Weight Model

- Challenge: Do not specify full distribution for sampling weight $\longrightarrow$ fully parametric bootstrap not apply
- Solution: Use MSE estimation procedure of Cho & Berg (2022)
  - Estimate M_{1i} as sample variance of $\theta_i^{(r)}$ $r = 1, \ldots, R$
  - Estimate M_{2i} from asymptotic normal distribution of parameter estimators

Beta-Prime Weight Model

- Method 1: Cho & Berg (2022)
- Method 2: fully parametric bootstrap (González-Manteiga et al. 2008)

Simulations: Set-Up

$$D = 50, \quad N_i = 200$$
$$y_{ij} \stackrel{ind}{\sim} \text{Bernoulli}(p_{ij})$$
$$\log\left(\frac{p_{ij}}{1-p_{ij}}\right) = -.2 + .7x_{ij} + u_i$$
$$u_i \stackrel{iid}{\sim} N(0, .25)$$

Parameters

- Mean $= N_i^{-1} \sum_{j=1}^{N_i} y_{ij}$
- Var $= (N_i - 1)^{-1} \sum_{j=1}^{N_i} (y_{ij} - N_i^{-1} \sum_{j=1}^{N_i} y_{ij})^2$
- Odds $= [\sum_{j=1}^{N_i} 1 - y_{ij}][\sum_{j=1}^{N_i} y_{ij}]^{-1}$

Alternative Procedures

- Inf: Proposed method
- Noninf: Proposed method with $\gamma = \alpha = \mathbf{0}$
- PL: Bayesian pseudo-likelihood (Parker et al. 2023)
  - Stan code provided on Github [1]

$$f(\boldsymbol{y}_{si} \mid \boldsymbol{u}, \boldsymbol{x}_{si}) = \prod_{j \in A_i} f(y_{ij} \mid u_i, \boldsymbol{x}_{ij})^{w_{ij}}$$
$$\boldsymbol{y}_{si} = \{y_{ij} : j \in A_i\}$$
$$\boldsymbol{u} = (u_1, \ldots, u_D)'$$

- MSE estimators
  - Inf-1: Cho & Berg (2022)
  - Inf-2: Fully parametric bootstrap
  - PL: posterior variance

[1] https://github.com/paparker/Unit_Level_Models/blob/master/Model_1.stan

Criteria

$$\text{AvMSE} = M^{-1}D^{-1}\sum_{i=1}^{D}\sum_{m=1}^{M}(\hat{\theta}_i^{(m)} - \theta_i^{(m)})^2$$

$$\text{Rel. Abs. Bias} = \frac{D^{-1}M^{-1}\sum_{i=1}^{D}|\sum_{m=1}^{M}(\hat{\theta}_i^{(m)} - \theta_i^{(m)})|}{D^{-1}M^{-1}\sum_{m=1}^{M}\sum_{i=1}^{D}\theta_i^{(m)}}$$

$$M\bar{S}E = M^{-1}D^{-1}\sum_{m=1}^{M}\sum_{i=1}^{D}M\hat{S}E_i^{(m)}$$

Case 1: Mean Weight Model

- PPS-systematic sample with

$$\pi_{ij} = \frac{10\exp(0.25y_{ij} + \delta_{ij})}{\sum_{k=1}^{N_i} \exp(0.25y_{ik} + \delta_{ik})}$$

	Rel. Abs. Bias			AveMSE			$\widehat{MSE}$	
	Inf	Noninf	PL	Inf	Noninf	PL	Inf	PL
Mean	0.53	11.63	1.29	0.90	1.17	1.00	0.89	0.96
Var	0.28	0.83	2.49	0.04	0.04	0.04	0.04	0.06
Odds	1.39	19.70	NA	38.52	44.09	NA	38.73	NA

Case 2: Beta Prime Weight Model

- PPS-systematic sample with

$$\pi_{ij} = \frac{\exp(-3 + 0.25 y_{ij})}{1 + \exp(-3 + 0.25 y_{ij})}.$$

	Rel. Abs. Bias			AveMSE			$\widehat{MSE}$		
	Inf	Noninf	PL	Inf	Noninf	PL	Inf-1	Inf-2	PL
Mean	0.86	11.75	1.55	0.84	1.12	0.91	0.76	0.78	0.88
Var	0.27	0.79	1.80	0.03	0.04	0.04	0.03	0.03	0.05
Odds	1.55	19.82	NA	35.77	41.17	NA	33.27	NA	NA

National Resources Inventory Application

- Longitudinal survey of agriculture and natural resources
  - We consider presence or absence of wetlands
  - We use data for 2012
- Multi-faceted sample design
  - Foundation sample
    - Stratified 2-stage sample
    - Observed 1982, 1987, 1992, 1997
  - Supplemented panel design
    - 2000-present
    - subsets of foundation sample observed each year
- State estimates published
  - We consider county estimation

National Resources Inventory Application

$$D = 21 \text{ counties in New Jersey}$$

$$y_{ij} = \begin{cases} 1 & \text{if wetland in 2012} \\ 0 & \text{otherwise} \end{cases}$$

Parameters: Mean, Var, Odds

Auxiliary Information

- Cropland data layer
- Sampled elements

$$x_{ij} = \begin{cases} 1 & \text{if CDL any kind of wetland} \\ 0 & \text{otherwise.} \end{cases}$$

- Regard a nonsampled location to represent 100 acres
  - $x_{ij} = 1$, $j = 1, \ldots, [A_{w,i}/100]$
  - $x_{ij} = 0$, $j = [A_{1,i}/100] + 1, \ldots, A_i$
  - $A_{w,i}$ = CDL wetland area of county i
  - A_i = area of county i

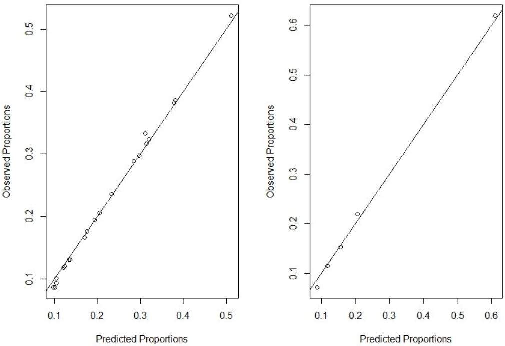
Sample Model for Application

$$
\begin{aligned}
y_{ij} &\overset{ind}{\sim} \text{Bernoulli}(p_{ij}) \\
\log\left(\frac{p_{ij}}{1-p_{ij}}\right) &= \beta_0 + \beta_1 x_{ij} + u_i \\
u_i &\overset{iid}{\sim} N(0, \sigma_u^2) \\
E_s(w_{ij} \mid y_{ij}, x_{ij}) &= \kappa_i \exp(\gamma_2 y_{ij} + \gamma_1 x_{ij})
\end{aligned}
$$

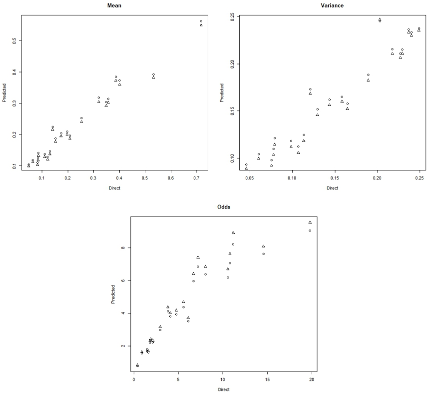
Model Parameter Estimates

	Est.	SE
β_0	-1.856	0.016
β_1	2.241	0.016
σ_u^2	0.211	0.004
γ_2	0.070	0.010

Evaluating Goodness of Fit



Predictions



- Circles: Proposed predictors against direct estimators.
- Triangles: Predictors assuming noninformative sampling against direct estimators.

Uncertainty Measures

- Average standard errors of direct estimators (Dir) and average root mean square errors of predictors (Pred).

	Dir	Pred
Mean	0.028	0.025
Variance	0.014	0.013
Odds	1.678	0.787

Summary and Discussion

- Developed a small area procedure to address three issues:
  - Exponential dispersion families, informative sampling, nonlinear parameters

Mean Weight vs. Beta-Prime Weight Model

- Mean weight model: requires fewer distributions, applicable if weights are 1
- Beta-prime weight model: allows straightforward MSE estimation

Comparison to Bayesian PL

- PL method uses relatively informative priors, and specification of more diffuse priors led to computational difficulties.
- Proposed method avoids the complicated problem of prior specifications
- Estimators for proposed method can be obtained using standard software

Extensions of the Basic Unit-Level Model

- Three extensions
  - Skewed data
  - Zero-inflated data
  - Informative sampling
- Key themes
  - Concepts of empirical best prediction
  - Importance of population-level covariate information
  - Generalizability to exponential dispersion families and nonlinear parameters

Thank You

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