An Estimation of Variance of Random Effects to Solve Multiple Problems in Small Area Estimation

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[Hirose and Lahiri, 2018, AoS]

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Outline

1. Small Area Estimation and Aggregated Level Model
2. Empirical Best Linear Unbiased Predictor under aggregated level model
3. A new variance component estimation for achieving desired properties
4. Monte Carlo simulation study
5. SAIPE data analysis
6. Conclusion

What is a small area estimation problem?

- Subpopulation inference is also very important, not only for the total population
- Direct estimates are constructed based only on each domain's sample data (Example: An estimation of Poverty rate: $\hat{p}_i^D = \sum_i w_{ij} y_{ij}$, where $y_{ij} \in \{0, 1\}$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n_i$.)

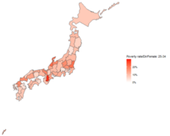

Figure: One example of Poverty mapping for Prefectures of Japan using Official microdata (Hirose and Oka in progress)

Note: The results in the analysis differ from published statistics in Japan.

- The small sample size may cause a large variation.
- refers to it as a Small area estimation problem

Aggregated level model for The Fay–Herriot Bayesian Model

There are two well-known kinds of explicit small-area models.

- Unit level model
- Aggregated level model

These models have played a critical role in the theory and practice of small-area estimation.

The unit-level model often requires unit-level data from confidential microdata.

Implementing an aggregated-level model does not tend to require confidential microdata compared with the unit-level model.

- Aggregate statistics are modeled; the chance of disclosing information about a given individual is low.
- Aggregate statistics are modeled; the relatively easier accessibility of aggregate statistics

The Fay-Herriot Bayesian Model

Fay and Herriot (1979)

For $i = 1, \cdots, m$,

Level 1: (Sampling model): $g(y_i)|\theta_i \sim N(\theta_i, D_i)$;

Level 2: (Linking model): $\theta_i \sim N(\boldsymbol{x}_i'\boldsymbol{\beta}, A)$

where

- m : number of small area;
- y_i : direct survey estimate;
- $g(y_i)$: transformed direct estimates using a smoothed monotone function g;
- θ_i : a true mean in transformed scale for area i;
- $\boldsymbol{x}_i$: p-vector of known auxiliary variables;
- D_i: *known* sampling variance of the direct estimate;
- The p-vector of regression coefficients $\boldsymbol{\beta}$ and model variance A are unknown.

Note: Hereafter, we focus on $g(\cdot) = (\cdot)$.

The Fay-Herriot Model As a Linear Mixed Model

The Fay-Herriot Bayesian model can be viewed as the following linear mixed model:

$$y = X\boldsymbol{\beta} + u + e,$$

where

- $X = (x_1', \ldots, x_m')'$
- $u = (u_1, \cdots, u_m)'$ and $e = (e_1, \cdots, e_m)'$ are independent with $u \sim N(0, AI)$, $e \sim N(0, D)$
- I: an identity matrix of dimension m;
- $D = diag(D_1, \cdots, D_m)$

We are interested in predicting

$$\boldsymbol{\theta} = (\theta_1, \cdots, \theta_m)' = X\boldsymbol{\beta} + u,$$

where $\theta_i = \boldsymbol{x}_i'\boldsymbol{\beta} + u_i,\ i = 1, \cdots, m.$

The Best Linear Unbiased Predictor (BLUP) of θ_i

When A is known, the following BLUP of θ_i is obtained by minimizing $MSE(\hat{\theta}_i)$ among all linear unbiased predictors of θ_i, where $MSE(\hat{\theta}_i) = E[(\hat{\theta}_i - \theta_i)^2]$ and E is the expectation with respect to Fay Herriot model:

$$\hat{\theta}_i^{BLUP} = (1 - B_i)y_i + B_i \boldsymbol{x}_i' \hat{\boldsymbol{\beta}},$$

where

- $B_i \equiv B_i(A) = \frac{D_i}{A+D_i}$
- $\hat{\boldsymbol{\beta}} \equiv \hat{\boldsymbol{\beta}}(A) = (X'V^{-1}X)^{-1}X'V^{-1}y$ where $V \equiv V(A) = diag(A + D_1, \cdots, A + D_m)$.

Empirical Best Linear Unbiased Predictor (EBLUP) of θ_i

Let $\hat{A}$ be a consistent estimator of model variance parameter A for large m.

An EBLUP of θ_i is given by

$$\hat{\theta}_i^{EBLUP} = (1 - \hat{B}_i) y_i + \hat{B}_i \boldsymbol{x}_i' \hat{\boldsymbol{\beta}}.$$

where

- $\hat{B}_i = \frac{D_i}{\hat{A}+D_i}$
- $\hat{\boldsymbol{\beta}} = \hat{\boldsymbol{\beta}}(\hat{A})$
- e.g., $\hat{A}$: PR estimator (Prasad and Rao, 1990), FH estimator (Fay and Herriot, 1979), ML, REML

Estimation of A: Likelihood-Based Methods

Profile Maximum Likelihood estimator (ML estimator)

$$\hat{A}_{ML} = \arg\max_{0 \le A < \infty} L_p(A|\boldsymbol{y}),$$

where

- $L_p(A, \boldsymbol{y}) = K|V|^{-1/2} \exp\{-\frac{1}{2} y' P y\}$, where K is a generic constant free from A;
- $P \equiv P(A) = V^{-1} - V^{-1}X(X'V^{-1}X)^{-1}X'V^{-1}$.

Residual Maximum Likelihood estimator (REML estimator)

$$\hat{A}_{RE} = \arg\max_{0 \le A < \infty} L_{RE}(A|y),$$

where $L_{RE}(A|y) = h_{RE}(A) L_p(A|\boldsymbol{y})$ with $h_{RE}(A) = |X'V^{-1}(A)X|^{-1/2}$.

Remarks: Over-shrinkage problem for an estimation of B_i; $\hat{B}_i = 1$. In such case, EBLUP gets over-shrinking to the regression estimator.

Estimation of A: Likelihood-Based Methods

Adjusted Maximum Likelihood Methods for avoiding zero estimates

- Li and Lahiri (2010)
  - The Li-Lahiri adjusted ML estimator (LL.ML):

$$\hat{A}_{LL.ML} = \underset{0<A<\infty}{\arg\max}\, h_{LL}(A)L_p(A|y), \text{ where } h_{LL}(A) = A.$$

  - The Li-Lahiri adjusted REML estimator (LL.RE):

$$\hat{A}_{LL.RE} = \underset{0<A<\infty}{\arg\max}\, h_{LL}(A)L_{RE}(A|y)$$

.

Remarks:

- That is, these methods provide the following property under mild regularity conditions; $0 < \inf_{i\geq 1} \hat{B}_i(\hat{A}) \leq \sup_{i\geq 1} \hat{B}_i(\hat{A}) < 1$.
- The MSE of $\hat{A}$ is all equivalent, up to order $O(m^{-1})$. The bias of $\hat{A}_{LL.ML}$ is of order $O(m^{-1})$ that is the same as the order of $\hat{A}_{ML}$. But the bias of $\hat{A}_{LL.RE}$ is?

Estimation of A: Likelihood-Based Methods

Adjusted Maximum Likelihood Methods for avoiding zero estimates

- Yoshimori and Lahiri (2014a, JMVA)
  - The Yoshimori-Lahiri adjusted ML estimator (YL.ML):

$$\hat{A}_{YL.ML} = \underset{0<A<\infty}{\arg\max}\, h_{YL}(A) L_p(A|\boldsymbol{y}), \text{ where } h_{YL}(A) = \arctan\left[\sum_{i}^{m}(1-B_i)\right]^{1/m}.$$

  - The Yoshimori-Lahiri adjusted REML estimator (YL.RE):

$$\hat{A}_{YL.RE} = \underset{0<A<\infty}{\arg\max}\, h_{YL}(A) L_{RE}(A|\boldsymbol{y})$$

Remarks:

- That is, these methods provide the following property under mild regularity conditions; $0 < \inf_{i\geq 1} \hat{B}_i(\hat{A}) \leq \sup_{i\geq 1} \hat{B}_i(\hat{A}) < 1$.
- Not only the MSE, but also these estimators of A enjoy the same asymptotic properties of ML and REML, up to the order of $O(m^{-1})$, respectively.

Mean Squared Error (MSE) of EBLUP

The MSE of BLUP under the Fay-Herriot model is derived as,

$$MSE_i^{BLUP} \equiv MSE(\hat{\theta}_i^{BLUP}) = g_{1i}(A) + g_{2i}(A),$$

where $g_{1i}(A) = \frac{AD_i}{A+D_i}$ and $g_{2i}(A) = \frac{D_i^2}{(A+D_i)^2} x_i'(X'V^{-1}X)^{-1}x_i$.

The MSE of EBLUP under the Fay-Herriot model is approximated for large m as,

$$MSE_i^{EBLUP} \equiv MSE[\hat{\theta}_i^{EBLUP}(\hat{A})] = g_{1i}(A) + g_{2i}(A) + g_{3i}(A) + o(m^{-1}),$$

where $g_{3i}(A) = \frac{2D_i^2}{(A+D_i)^3 tr[V^{-2}]}$ and $\hat{A} \in \{\hat{A}_{ML},\ \hat{A}_{RE},\ \hat{A}_{LL.ML}, \hat{A}_{LL.RE}, \hat{A}_{YL.ML}, \hat{A}_{YL.RE}\}$.

still depends on an unknown parameter...

A second-order unbiased estimator of Mean Squared Error (MSE) for EBLUP

Definition: A second-order unbiased MSE estimator for true MSE, $\widehat{MSE}$

$\widehat{MSE}$ is satisfying that $E[\widehat{MSE} - MSE] = o(m^{-1})$ for large m.

The naive estimator: plugged $\hat{A}$ into MSE of BLUP does not satisfy.

$$E\left[MSE_i(\hat{\theta}_i^{BLUP}(A))\Big|_{A=\hat{A}} - MSE\right] = O(m^{-1}), \qquad (1)$$

where $\hat{A} \in \{\hat{A}_{ML},\ \hat{A}_{RE},\ \hat{A}_{LL.ML}, \hat{A}_{LL.RE}, \hat{A}_{YL.ML}, \hat{A}_{YL.RE}\}$.

A second-order unbiased estimator of Mean Squared Error (MSE) for EBLUP

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$\widehat{MSE}$ is satisfying that $E[\widehat{MSE} - MSE] = o(m^{-1})$ for large m.

Bias correction terms are required:

1. **Taylor linearization** (Prasad and Rao, 1990; Datta and Lahiri, 2000; Datta et al., 2004; Das et al., 2004;Li and Lahiri, 2010; Yoshimori and Lahiri, 2014b)
   Using $\hat{A}_{ML}$ $/\hat{A}_{LL.ML}$ $/\hat{A}_{LL.RE}/\hat{A}_{YL.ML}$
   $$\widehat{MSE}_i \equiv \widehat{MSE}_i[\hat{\theta}_i(\hat{A})] = g_{1i}(\hat{A}) + g_{2i}(\hat{A}) + 2g_{3i}(\hat{A}) - b(\hat{A})\hat{B}_i^2,$$
   where $b(A)$ is a bias of $\hat{A}$, up to the order $O(m^{-1})$.

   Using $\hat{A}_{RE}$/ $\hat{A}_{YL.RE}$
   $$\widehat{MSE}_i \equiv \widehat{MSE}_i[\hat{\theta}_i(\hat{A})] = g_{1i}(\hat{A}) + g_{2i}(\hat{A}) + 2g_{3i}(\hat{A}).$$

2. **Jackknife method** (Jiang, Lahiri and Wan, 2002; Chen and Lahiri, 2008): In this presentation, we won't focus on the Jackknife method.

Parametric Bootstrap estimator of Mean Squared Error for EBLUP

3. **Parametric Bootstrap method [Single]** (Butar and Lahiri, 2003)

$$\widehat{MSE}_i^{BL} \equiv \widehat{MSE}_i[\hat{\theta}_i(\hat{A})] = 2[g_{1i}(\hat{A}) + g_{2i}(\hat{A})] - \frac{1}{B}\sum_{b=1}^{B}[g_{1i}(\hat{A}^{(b)}) + g_{2i}(\hat{A}^{(b)})] + \frac{1}{B}\sum_{b=1}^{B}[\hat{\theta}_i(y, \hat{\beta}^{(b)}, \hat{A}^{(b)}) - \hat{\theta}_i(y, \hat{\beta}^{(b)}, \hat{A}^{(b)})]^2.$$

Remark

They could be negative MSE estimates due to their bias corrections.

Parametric Bootstrap estimator of Mean Squared Error for θ_i

4. **Parametric Bootstrap method [Double]**
   (Hall and Maiti, 2006; Chatterjee and Lahiri, 2007).
   e.g., one of the estimators of Hall and Maiti (2006) is given by the following;

$$\widehat{MSE}_i^{HM1} = \begin{cases} 2\hat{u} - \hat{v} & (\hat{u} \geq \hat{v}) \\ \exp[-(\hat{v} - \hat{u})/\hat{v}]\hat{u} & (\hat{u} < \hat{v}) \end{cases}$$

where $\hat{u} = \frac{1}{B}\sum_{b=1}^{B}\left[\hat{\theta}_i^{(b)}(y^{(b)}, \hat{\beta}^{(b)}, \hat{A}^{(b)}) - \theta_i^{(b)}\right]^2$,
$\hat{v} = \frac{1}{B}\sum_{b=1}^{B}\left[\frac{1}{C}\sum_{c=1}^{C}\left[\hat{\theta}_i^{(bc)}(y^{(bc)}, \hat{\beta}^{(bc)}, \hat{A}^{(bc)}) - \theta_i^{(bc)}\right]^2\right]$.

These MSE estimators are strictly positive, but the double bootstrap method is more computer-intensive than the single bootstrap method. And not sure about the second-order unbiasedness (Jiang et al., 2016)

Research Question

What are desired properties?

- For θ_i, We need to focus on estimating the shrinkage factor B_i, rather than that of A.
- We wish to protect EBLUP from over-shrinking to the regression estimator.
- There is also a desire to use a simple second-order unbiased MSE estimator to maintain the MSE estimator's strict positivity for practical users.

Research Question

What are desired properties?

Desired properties

1. Obtain a second-order unbiased estimator of B_i;
   $E(\hat{B}_i) = B_i + o(m^{-1})$ in maintaining equivalent identical variance of other likelihood-based methods, up to the order $O(m^{-1})$.
2. $0 < \inf_{m \geq 1} \hat{B}_i \leq \sup_{m \geq 1} \hat{B}_i < 1$ for protecting EBLUP from over-shrinking to the regression estimator;
3. Obtain a simple second-order unbiased Taylor series MSE estimator of EBLUP without any bias correction; that is, $\widehat{MSE}_i = g_{1i}(\hat{A}) + g_{2i}(\hat{A}) + g_{3i}(\hat{A})$;
4. Produce a strictly positive second-order unbiased single parametric bootstrap MSE estimator without bias correction.

$$\widehat{MSE}_i^{PB} = E_*[(\hat{\theta}_i^{EBLUP}(\hat{A}^*, y^{(*)}) - \theta_i^*)^2];$$

Research Question

Then, can we achieve these four desired properties simultaneously?

To address such an issue, we propose an area-specific estimator of A, say $\hat{A}_i$, that simultaneously satisfies these multiple desirable properties under certain mild regularity conditions.

A new adjusted maximum likelihood estimator of A

The residual maximum likelihood estimator of A is defined as:

$$\hat{A}_{RE} = \arg\max_{0 \le A < \infty} L_{RE}(A|y).$$

Note that $\hat{A}_{RE}$ does not satisfy any of the four desirable properties.

To find a likelihood-based estimator of A that satisfies all the four desirable properties, we start by setting up a general adjusted maximum likelihood estimator of A defined as:

$$\hat{A}_i = \arg\max_{0 < A < \infty} h_i(A) L_{RE}(A), \quad (2)$$

where $h_i(A)$ is not specified.

A new adjusted maximum likelihood estimator of A

We first find the adjustment factor $h_i(A)$ that satisfies Property 1.
Under the mild regularity conditions, we have, for large m,

$$E(\hat{B}_i) = B_i + \left[\frac{\partial B_i}{\partial A}\frac{\partial \log h_i(A)}{\partial A} + \frac{1}{2}\frac{\partial^2 B_i}{\partial A^2}\right]\frac{2}{tr[V^{-2}]} + o(m^{-1}).$$

Thus, Property 1 is satisfied if we have

$$\frac{\partial B_i}{\partial A}\frac{\partial \log h_i(A)}{\partial A} + \frac{1}{2}\frac{\partial^2 B_i}{\partial A^2} = 0.$$

Thus, an adequate adjustment factor is given by

$$\mathbf{h_{i0}(A) = (A + D_i)}.$$

This adjustment factor is indeed the unique solution, up to the order of $O(1)$ for large m.

A new adjusted maximum likelihood estimator of A

The resulting estimator is given by,

$$\hat{A}_i = \arg\max_{0<A<\infty} \tilde{h}_0(A) L_{RE}(A).$$

Interestingly, it turns out that such an adjusted maximum likelihood estimator also satisfies Properties 3 and 4.

"$\hat{A}_i$ satisfy Property 1, 3, 4 but not Property 2..."

We propose our final estimator of A for $m > p + 2$ as:

$$\hat{A}_{i;MG} = \arg\max_{0<A<\infty} \tilde{h}_i(A) L_{RE}(A),$$

where $\tilde{h}_i(A) = h_+(A) h_{i0}(A)$ with the additional adjustment $h_+(A)$ satisfying several conditions.
The choice of $h_+(A)$ is generally not unique. One can use the choice h_{YL} given in Yoshimori and Lahiri (2014a, JMVA).

A new adjusted maximum likelihood estimator of A

Theorem 1

Under some mild regularity conditions, we have, for large m,

$$(i) E[\hat{B}_{i;MG} - B_i] = o(m^{-1});\ Var(\hat{B}_{i;MG}) = \frac{2D_i^2}{(A+D_i)^4 tr[V^{-2}]} + o(m^{-1});$$

$$(ii)\ 0 < inf_{m\geq 1}\hat{B}_{i;MG} \leq sup_{m\geq 1}\hat{B}_{i;MG} < 1,\ for\ m > p+2;$$

$$(iii) E[\widehat{MSE}_{i;MG} - MSE_i(\hat{\theta}^{EB}_{i;MG})] = o(m^{-1});$$

$$(iv)\ E[\widehat{MSE}^{PB}_{i;MG} - MSE_i(\hat{\theta}^{EB}_{i;MG}) = o(m^{-1}),$$

where

$$\hat{B}_{i;MG} = B_i(\hat{A}_{i;MG});\quad \hat{\theta}^{EB}_{i;MG} = \hat{\theta}^{BLUP}_i(\hat{A}_{i;MG});$$

$$\widehat{MSE}_{i;MG} = g_{1i}(\hat{A}_{i;MG}) + g_{2i}(\hat{A}_{i;MG}) + g_{3i}(\hat{A}_{i;MG});$$

$$\widehat{MSE}^{PB}_{i;MG} = E_*[(\hat{\theta}_i(\hat{A}^*_{i;MG}, y^{(*)}) - \theta^*_i)^2].$$

Our approach also ensures the important dual properties of the MSE estimator — second-order unbiasedness and strict positivity.

Simulation set-up

We considered the SAIPE program of the U.S. Census Bureau to estimate the percentages of school-age children in poverty for the fifty states and the District of Columbia. (http://www.census.gov/did/www/saipe/about/index.html, Bell et al. 2015)
To compare the performances in using $\hat{A}_{RE}$ with that of $\hat{A}_{i;MG}$, we use x_i and D_i from the same SAIPE data set for the 1992 year, considered by Bell (1999).

- The 15 areas correspond to states with the largest sampling variances D_i.
- $A = 15.94$ which is the median of D_i for the 15 states.
- β: The weighted least squared estimate of β from the real data, including all 50 states and DC. ($p = 5$)

Result 1: RB and RRMSE of $\hat{B}_i$

$$\text{RB of } \hat{B}_i : \frac{\mathrm{E}(\hat{B}_i - B_i)}{B_i} \times 100;$$

$$\text{RRMSE of } \hat{B}_i : \frac{\sqrt{\mathrm{MSE}(\hat{B}_i)}}{B_i} \times 100.$$

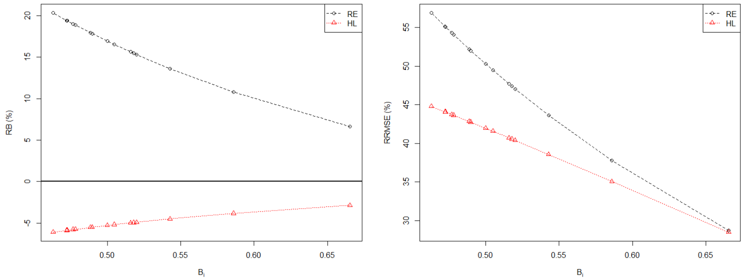
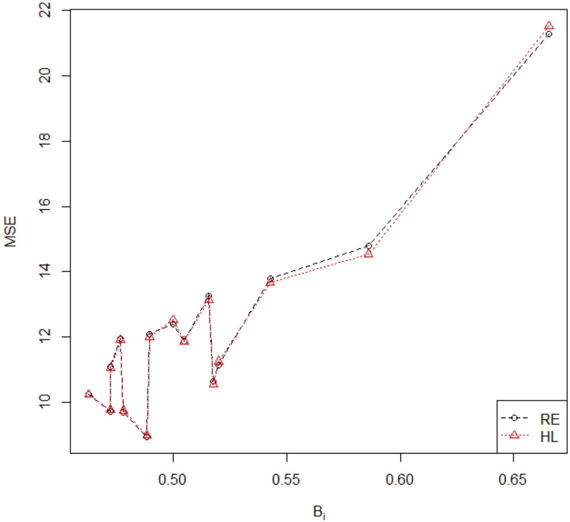


Figure: RB and RRMSE of $\hat{B}_i$

Result 2: MSE of EBLUP



We also report simulated RBs and RRMSE of different MSE estimators of EBLUPs that use $\hat{A}_{RE}$ and ours.

1. Naive MSE estimator (naive.RE): $g_{1i}(\hat{A}_{RE}) + g_{2i}(\hat{A}_{RE})$;
2. Single parametric bootstrap MSE estimator (PB.RE):
$$E_*[(\hat{\theta}_i(\hat{A}^*_{RE}, y^{(*)}) - \theta^*_i)^2];$$
3. DL.RE: $g_{1i}(\hat{A}_{RE}) + g_{2i}(\hat{A}_{RE}) + 2g_{3i}(\hat{A}_{RE})$;
4. Taylor.HL: the proposed Taylor series MSE estimator,
$$\widehat{MSE}_{i;MG} = g_{1i}(\hat{A}_{i;MG}) + g_{2i}(\hat{A}_{i;MG}) + g_{3i}(\hat{A}_{i;MG});$$
5. PB.HL: our proposed single parametric bootstrap MSE estimator,
$$\widehat{MSE}^{PB}_{i;MG} = E_*[(\hat{\theta}_i(\hat{A}^*_{i;MG}, y^{(*)}) - \theta^*_i)^2];$$
6. PB.BL:
$$2\{g_{1i}(\hat{A}_{RE}) + g_{2i}(\hat{A}_{RE})\} - E_*[g_{1i}(\hat{A}^*_{RE}) + g_{2i}(\hat{A}^*_{RE})] + E_*[\{\hat{\theta}^*_i(y_i, \hat{A}^*_{RE}, \hat{\beta}(\hat{A}^*_{RE}, y)) - \tilde{\theta}^*_i(y, \hat{A}_{RE}, \hat{\beta}(\hat{A}_{RE}, y_i))\}^2].$$

Result 3

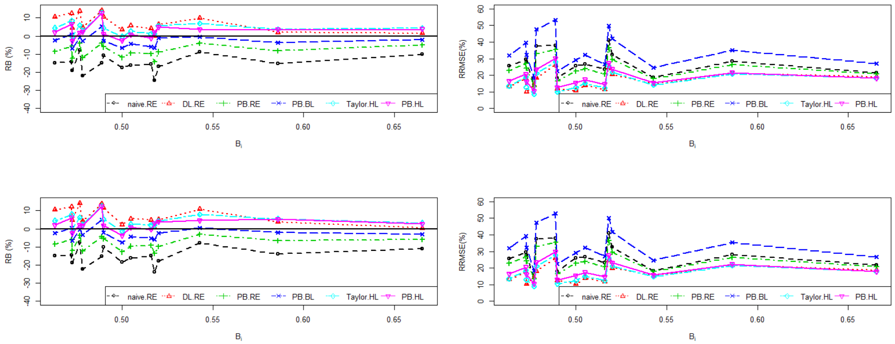


Figure: RB and RRMSE of MSE estimators for MSE of EB using REML(above) and HL(bottom); states are arranged in decreasing order of the sampling variances

Data Analysis

We consider 1992 and 1993 SAIPE data. In 1992, the REML estimate of A was zero, while in 1993, it was positive.
For this application, the small areas are 50 states and the District of Columbia of the United States, so $m = 51$.

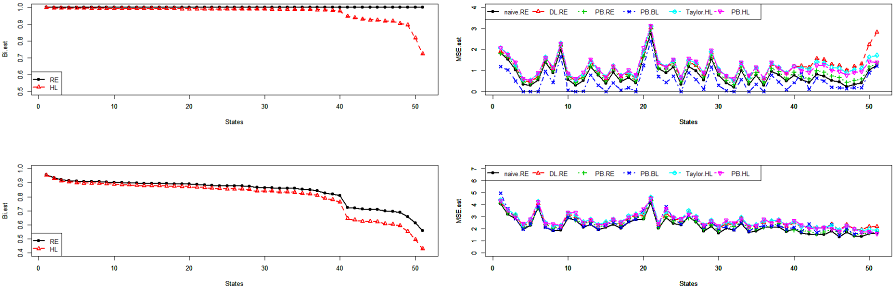


Figure: Estimates of B_i and MSE using all SAIPE data for 1992 (above) and 1993(bottom) year; states are arranged in decreasing order of the sampling variances

Conclusion

- Explanation of the basic EBLUP theory
- Proposed new variance estimator for achieving multiple goals simultaneously.
- Overall, we demonstrated that our proposed method offers reasonable results

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Thank you for your listening!